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**Abstract:** *It is known from the classical theory of Lie algebras that any finite-dimensional Lie algebra over a field of characteristic zero can be decomposed into a semidirect sum of its maximal solvable ideal and its semisimple subalgebra. Similarly, finite-dimensional Leibniz algebras also decompose into a semidirect sum of their maximal solvable ideal and a semisimple Lie algebra. The study of solvable algebras with nilradicals of special types is related to various models in physics. Thus, similar to the case of Lie algebras, studying Leibniz algebras with given nilradicals is an important problem.*

*It should be noted that nilpotent Lie algebras are a special type of solvable algebras. Since describing nilpotent Lie algebras is an enormous task, their study must be conducted with additional constraints. In particular, one of the main constraints when studying nilpotent algebras is the limitation on the nilpotency index. It is important to emphasize that the maximum nil-index for a Lie algebra coincides with the dimension of the algebra itself, and such algebras are called filiform algebras.*

*Although Leibniz filiform algebras in the class of nilpotent algebras have relatively simple restrictions, they possess a complex structure, which is conveniently studied by imposing a gradation condition. The effectiveness of the maximal grading lies in the fact that it provides the most precise information about the structural constants in the multiplication table of the algebra.*

*"Currently, the structural theory of Lie algebras holds an important place in the study of other branches of group theory. Numerous articles have been published on these algebras, and their structural theory has been studied.*

*As is known from the classical theory of finite-dimensional Lie algebras, the study of finite-dimensional Lie algebras is reduced to the study of nilpotent Lie algebras. The class of Lie algebras with maximal nilindex is an important part of the class of nilpotent Lie algebras.*

**Keywords:** *Lie algebras,  $n$ -dimensional, 2-filiform, natural graduated, maximal length.*

## INTRODUCTION

It is known that for Lie algebras, the maximal nilindex coincides with the algebra dimension, and such algebras are called filiform algebras. Although filiform Lie algebras are a relatively simple part of nilpotent algebras, they have sufficiently complex properties. Therefore, it is more convenient to study them using graduation conditions, and various types of gradings are used in this regard, such as natural gradings, maximal length gradings, and gradings corresponding to a given filtration.

According to the Levi-Malcev theorem, any finite-dimensional Lie algebra can be expressed as a direct sum of a semi-simple Lie algebra and the semi-direct sum of its solvable radical. This result implies that the study of solvable algebras is of significant importance in the structural theory of finite-dimensional Lie algebras.

In the classification of finite-dimensional Lie algebras and the study of their structural theory, naturally graded nilpotent Lie algebras have special importance. The concept of maximal length grading was introduced by Yu. Hakimjanov, and it provides complete information about the subspaces of the algebra and the products between them.

Natural gradings and maximal length gradings are closely related to each other. The study of finite-dimensional Lie algebras is reduced to the study of nilpotent Lie algebras.

Filiform Lie algebras are considered to have the highest nilindex, and all  $n$ -dimensional filiform Lie algebras of lengths  $n$  and  $n-1$  have been fully classified in the works of X.R. Gomes and other scholars. 2-filiform Lie algebras with nilindex  $n-2$  are considered to be  $n$ -dimensional nilpotent Lie algebras, and the maximal length  $n$ -dimensional Lie algebras have also been fully classified in the works of X.R. Gomes. This master's thesis focuses on the classification of 2-filiform Lie algebras with length  $n-1$ . Let me know if you need further adjustments or clarifications!

The gradations of algebras are very important to investigation of the structural theory and properties of those algebras. The graduation which length is equal to the dimensional of the algebra, is called the maximum graduation. Such gradations are useful for the investigation of cohomologies for the considered algebras because they induce a corresponding gradation of the group of cohomologies. The various papers are devoted to investigation of Leibniz algebras of maximum length or length an algebras is less than the one from maximum.

In this work, we consider some class of naturally graded quasi-filiform Leibniz algebras of length  $n - 1$ .

To determine the length of the algebras, first, they were divided into naturally graded spaces, and their lengths were determined. If the grading length was not maximal, an attempt was made to bring it to the maximal length by changing the basis, and the desired result was achieved.

**Definition 1.** An algebra  $(L,[-,-])$  over a field  $F$  is called Lie algebra if for any  $x, y, z \in L$  the so-called Leibniz identity:

$$[x, [y, z]] = [[x, y], z] + [[y, z], x] + [[x, z], y] = 0$$

holds.

For an arbitrary Lie algebra  $L$  we define the lower central series:

$$L^1 = L, L^{k+1} = [L^k, L], k \geq 1.$$

**Definition 2.** An  $n$ -dimensional Lie algebra  $L$  is called quasi-filiform if  $L^{n-2} \neq 0, L^{n-1} = 0$ .

Let  $L$  be a  $Z$ -graded Leibniz algebra with a finite number of non zero subspaces, i.e.  $L = \bigoplus_{i \in Z} V_i$ , where  $[V_i, V_j] \subseteq V_{i+j}$  for any  $i, j \in Z$ . We say that a Leibniz algebra  $L$  admits a *connected gradation* if  $L = V_{k_1} \oplus V_{k_2} \oplus \dots \oplus V_{k_t}$  where each  $V_i$  is non-zero for  $k_1 \leq i \leq k_t$ .



The number of subspaces  $l(\oplus L) = k_t - k_1 + 1$  is called the length of the gradation. The length  $l(L)$  of a Leibniz algebra  $L$  is defined as

$$l(L) = \{ \max(\oplus L) = k_t - k_1 + 1 | L = V_{k_1} \oplus V_{k_2} \oplus \dots \oplus$$

$V_{k_t}$  is a connected gradation }.

An algebra  $L$  is called maximum length, if  $l(L) = \dim(L)$ .

**Definition 3.** For a given nilpotent Lie algebra  $L$ , put  $L_i = L^i / L^{i+1}$ ,  $1 \leq i \leq n - 1$  and  $gr(L) = L_1 \oplus L_2 \oplus \dots \oplus L_{n-1}$ . Then  $[L_i, L_j] \subseteq L_{i+j}$  and we obtain the graded algebra  $gr(L)$ . If  $gr(L)$  and  $L$  are isomorphic, then we say that an algebra  $L$  is naturally graded.

The filiform algebras characterized by the maximal  $n - 1$  nilindex were studied by Vern. When the dimension of the algebra is even, algebras of type  $L_n$  and  $O_n$  are considered, while for odd dimensions, only algebras of type  $L_n$  are presented. The  $L_n$  and  $O_n$  algebras  $(x_0, x_1, \dots, x_{n-1})$  are determined by the following multiplication in the changed basis.

$$L_n: [x_0, x_i] = x_{i+1}, \quad 1 \leq i \leq n - 2$$

$$Q_n: \begin{cases} [x_0, x_i] = x_{i+1}, & 1 \leq i \leq n - 2 \\ [x_i, x_{n-1-i}] = (-1)^{i-1} x_{n-1}, & 1 \leq i \leq q - 1, n = 2q \end{cases}$$

The  $Q_n$  filiform algebra does not form a grading with  $n$  part spaces. Furthermore, there are other  $n$ -dimensional filiform Lie algebras that differ from  $L_n$ . Algebras like  $L_n$  and  $Q_n$  provide essential information about the structure of the family of filiform algebras when studying the multiplicity of nilpotent algebras.

In addition to  $L_n$ , there are other filiform Lie algebras that include maximal-length gradings. Let us denote a by the whole part. The algebras given through  $R_n$  and  $W_n$  in , and through  $K_n$  and  $Q_n$  in , are defined by the following multiplication in a homogeneous basis:

$$R_n: \begin{cases} [x_0, x_1] = x_{i+1}, & 1 \leq i \leq n - 2 \\ [x_i, x_j] = x_{2+j}, & 2 \leq j \leq j \leq n - 3 \end{cases}$$

$$W_n: \begin{cases} [x_0, x_1] = x_{i+1}, & 1 \leq i \leq n - 2 \\ [x_i, x_j] = \frac{6(i-1)!(j-1)!(j-1)}{(i+j)!} x_{i+j+1}, \\ 1 \leq i \leq j \leq \left\lfloor \frac{n-3}{2} \right\rfloor, j \leq n - 2 - i \end{cases}$$

$$K_n: \begin{cases} [x_0, x_1] = x_{i+1}, & 1 \leq i \leq n - 2 \\ [x_i, x_{2\lfloor \frac{n-2}{2} \rfloor - i}] = (-1)^{i-1} x_{2\lfloor \frac{n-2}{2} \rfloor} \\ 1 \leq i \leq \left\lfloor \frac{n-4}{2} \right\rfloor, \\ [x_i, x_{2\lfloor \frac{n-2}{2} \rfloor - i}] = (-1)^{i-1} 2 \left\lfloor \frac{n-2}{2} \right\rfloor - i x_{2\lfloor \frac{n-2}{2} \rfloor + 1}, \\ 1 \leq i \leq \left\lfloor \frac{n-4}{2} \right\rfloor, \\ [x_i, x_{n-2-i}] = \frac{1}{2} (-1)^i (i-1)(n-3-i) \alpha x_{n-1} \\ 1 \leq i \leq \left\lfloor \frac{n-3}{2} \right\rfloor. \end{cases}$$



If  $n$  is even, then  $\beta = 0$ , and if  $n$  is odd, then  $\beta = 1$ .  $l(Rn) = n$ ;  $l(Wn) = n$ ;  $l(Kn)$  and It is easy to verify that

$$(Q'n) = 0$$

Furthermore, these algebras are the only Lie algebras with maximal length equal to their dimensions.

First, we need to determine the structure of the maximal-length quasi-filiform algebras. In Vern's work, to obtain naturally graded Lie algebras, there exists a basis of the  $n$ -dimensional  $g$  filiform Lie algebra of the form  $(x_0, x_1, x_2, \dots, x_{n-1})$ .

If we consider the family of nilpotent Lie algebras and search for a graded algebra of a certain length, we cannot assume that a basis expressing the algebras will always exist. This will be the first problem in classifying such Lie algebras.

Thus, we will first obtain a homogeneous basis that allows us to determine the structure of the family of Lie algebras. Therefore, in the quasi-filiform case, the number of part spaces is  $n - 1$ , and the case where the number of part spaces is  $n$  (i.e., the maximal-length grading) is of interest to us.

In this section, it is established that the lengths of the naturally graded odd and even-dimensional quasi-filiform Lie algebras are equal to  $n - 1$ .

Proposition 1. The length of the following odd-dimensional quasi-filiform Lie algebra is equal to  $n - 1$ .

$$Q_{n-1} + C, (n \geq 7, n - \text{odd}): \begin{cases} [e_0, e_1] = e_{i+1}, & 1 \leq i \leq n - 3 \\ [e_i, e_{n-2-i}] = (-1)^{i-1} e_{n-2}, & 1 \leq i \leq \frac{n-3}{2} \end{cases}$$

This algebra has two generating elements, which we denote as  $x_s$  and  $x_t$  for some values of  $s$  and  $t$ . These elements are linear combinations of the basis elements of the  $Q$  algebra.

$$\begin{aligned} x_s &= \sum a_i e_i, & x_t &= \sum a_i e_i, \\ a_1 b_{n-1} - a_{n-1} b_1 &\neq 0 \\ a_1 b_{n-1} &= 1 \end{aligned}$$

We take the following multiplications.

$$\begin{aligned} [x_s, x_t] &= e_2 + b_2 e_3 + b_3 e_4 + b_4 e_5 + \dots + b_{n-3} e_{n-2} - a_1 b_0 e_2 + a_1 b_{n-3} e_{n-2} \\ &\quad - a_2 b_{n-4} e_{n-2} + a_3 b_0 e_4 + a_3 b_{n-5} e_{n-2} - a_4 b_0 e_5 - a_4 b_{n-6} e_{n-2} + a_5 b_0 e_6 \\ &\quad + a_5 b_{n-7} e_{n-2} - \dots - a_{n-3} b_0 e_{n-2} \\ &= (1 - a_1) e_2 + (b_2 - a_2 b_0) e_3 + (b_3 - a_3 b_0) e_4 + \dots + (b_{n-3} - a_{n-3} b_0) e_{n-2} \\ &\quad + (a_1 b_{n-3} - a_2 b_{n-4}) e_{n-2} \\ [[x_s, x_t] x_t] &= -(1 - a_1 b_0) b_0 e_3 + (1 - a_1 b_0) b_{n-4} e_{n-2} + (b_2 - a_2 b_0) b_{n-5} e_4 \\ &\quad - (b_3 - a_3 b_0) b_{n-6} e_{n-2} + \dots + (b_{n-4} - a_{n-4} b_0) \frac{b_{n-3}}{2} e_{n-2} - (1 - a_1 b_0) b_0 e_3 \\ &\quad + (b_2 - a_2 b_0) e_4 + \dots + ((1 - a_1 b_0) b_{n-4} + (b_3 - a_3 b_0) b_{n-6} \\ &\quad + (b_{n-4} - a_{n-4} b_0) b_0 + \dots + (b_{n-4} - a_{n-4} b_0) \frac{b_{n-3}}{2}) e_{n-2} \end{aligned}$$

In general, the multiplication takes the following form

$$\begin{aligned} [\dots [x_s, x_t] \dots x_t] &= (1 - a_1 b_0) b_0^{i-1} e_i + (*) e_{i+1} + (*) e_{i+2} + \dots + (*) e_{n-2} \\ &3 \leq i \leq n - 2 \end{aligned}$$



Case 1. Let be

$$b_0 a_1 - 1 + 0$$

$$y_i \in V_{k_s+(n-3)k_t}, 1 \leq i \leq n-1$$

And from the dependence of the grading, we obtain the equality  $kt = l$ .

Let's say

$$y_1 \in V_{k_s}, y_2 \in V_1, y_3 \in V_{k_s}, \dots, y_{n-1} \in V_{k_s+n-3}$$

In case  $k_s = 2$ , it would be appropriate

$$L = V_1 + V_2 + V_3 + \dots + V_{n-1}$$

We will consider the following products:

$$[x_s, x_t] = -(e_2 + b_2 e_3 + b_3 e_4 + b_4 e_5 + \dots + b_{n-3} e_{n-2} - a_1 b_0 e_2 + a_1 b_{n-3} e_{n-2} - a_2 b_{n-4} e_{n-2} + a_3 b_0 e_4 + a_3 b_{n-5} e_{n-2} - a_4 b_0 e_5 - a_4 b_{n-6} e_{n-2} + a_5 b_0 e_6 + a_5 b_{n-7} e_{n-2} - \dots - a_{n-3} b_0 e_{n-2})$$

$$= (1 - a_1) e_2 + (b_2 - a_2 b_0) e_3 + (b_3 - a_3 b_0) e_4 + \dots + (b_{n-3} - a_{n-3} b_0) e_{n-2} + (a_1 b_{n-3} - a_2 b_{n-4}) e_{n-2}$$

$$[[x_s, x_t], x_t] = -(1 - a_1 b_0) b_0 e_3 + (1 - a_1 b_0) b_{n-4} e_{n-2} + (b_2 - a_2 b_0) b_{n-5} e_4 - (b_3 - a_3 b_0) b_{n-6} e_{n-2} + \dots + (b_{n-4} - a_{n-4} b_0) b_0 e_{n-2} + (b_{n-4} - a_2 b_{n-4}) b_0 e_{n-2} + (b_{n-4} - a_{n-4} b_0) \frac{b_{n-3}}{2} e_{n-2} - (1 - a_1 b_0) b_0 e_3 + (b_2 - a_2 b_0) e_4 + \dots + ((1 - a_1 b_0) b_{n-4} + (b_3 - a_3 b_0) b_{n-6} + (b_{n-4} - a_{n-4} b_0) b_0 + \dots + (b_{n-4} - a_{n-4} b_0) b_{(n-3)/2}) e_{n-2}$$

Our grading will take the following form:

$$y_1 \in V_1, y_2 \in V_2, y_3 \in V_3, \dots, y_{n-1} \in V_{n-1}$$

We will check the conditions of the grading:

$$y_1 = e_1$$

$$y_2 = e_0 + a_1 + e_1$$

$$y_3 = -e_2$$

$$y_4 = e_3$$

$$y_i = (-1)^i e_{i-1}, 3 \leq i \leq n-1$$

$$[y_1 y_1] = 0, \quad V_2 V_2 \subset V_4 \quad [y_2 y_3] = y_4, \quad V_1 V_3 \subset V_4$$

$$[y_1 y_2] = y_3, \quad V_2 V_1 \subset V_3 \quad [y_3 y_1] = 0, \quad V_3 V_2 \subset V_5$$

$$[y_1 y_3] = 0, \quad V_2 V_3 \subset V_5 \quad [y_3 y_2] = -y_4, \quad V_3 V_1 \subset V_4$$

$$[y_2 y_1] = -y_3, \quad V_1 V_2 \subset V_3 \quad [y_3 y_3] = 0, \quad V_3 V_3 \subset V_6$$

$$[y_2 y_2] = 0, \quad V_1 V_1 \subset V_2$$

That is, the following is appropriate:

$$V_i V_2 \subset V_{i+2} \quad V_3 = \langle y_3 \rangle \quad \dots \quad V_i = \langle y_i \rangle \quad 3 \leq i \leq n-1$$

So, the length of the maximal grading is  $n-1$ .

Proposition 2. The length of the following L algebra is equal to  $n-1$ .  $L_{(n,r)} (n \geq$

$$5, r - \text{odd}, 3 \leq x \leq 2 \lfloor \frac{(n-1)}{2} \rfloor - 1$$

$$\begin{cases} [e_0, e_1] = e_{n+1}, 1 \leq i \leq n-3 \\ [e_i, e_{r-i}] = (-1)^{i-1} e_{n+1}, 1 \leq i \leq (r-1)/2 \end{cases}$$



**Proof.** This algebra has two generators, which we denote as  $x_s$  and  $x_t$  for some  $s$  and  $t$ . We will introduce the following commutation relations and determine the products.

$$x_s = e_0 + a_1e_1 + a_2e_2 + a_3e_3 + \dots + a_{n-3}e_{n-3}$$

$$x_t = e_0 + b_1e_1 + b_2e_2 + b_3e_3 + \dots + b_{n-3}e_{n-3}$$

Without loss of generality, we take  $a_1 = b_0 = 1$

$$y_1 = x_s, y_2 = x_t$$

$$\begin{aligned} [x_s, x_t] &= e_2 + b_2e_3 + b_3e_4 + b_4e_5 + \dots + b_{n-3}e_{n-2} - a_1b_0e_2 + a_1b_6e_3 - a_2b_5e_{n-1} \\ &\quad + a_3b_0e_4 + a_3b_4e_{n-1} - a_4b_0e_5 - a_4b_3e_{n-1} + a_5b_0e_6 + a_5b_2e_{n-1} - \dots \\ &\quad - a_{n-3}b_0e_{n-2} \\ &= (1 - a_1)e_2 + (b_2 - a_2b_0)e_3 + (b_3 - a_3b_0)e_4 + \dots + (b_{n-3} - a_{n-3}b_0)e_{n-2} \\ &\quad + (a_1b_6 - a_2b_5 + a_3b_4 + a_4b_3 - a_5b_2 - a_6 \dots)e_{n-2} \end{aligned}$$

$$\begin{aligned} y_4 &= [[x_s, x_t], x_t] = \\ &= -(1 - a_1b_0)b_0e_3 + (1 - a_1b_0)b_5e_{n-1} + (b_2 - a_2b_0)b_0e_4 \\ &\quad - (b_2 - a_2b_0)b_4e_{n-1} - (b_3 - a_3b_0)b_0e_5 - (b_3 - a_3b_0)b_0e_{n-1} + \dots \\ &\quad + (b_{n-4} - a_{n-4}b_0)b_0e_{n-2} \\ &= -(1 - a_1b_0)b_0e_3 - (b_2 - a_2b_0)e_4 - (b_3 - a_3b_0)b_0e_5 - (b_4 - a_4b_0)b_0e_6 \\ &\quad - \dots - (b_{n-4} - a_{n-4}b_0)b_0e_{n-2} + \dots + (b_2 - a_2b_0)b_4 - (b_3 - a_3b_0)b_3 \\ &\quad + \dots)e_{n-1} \end{aligned}$$

$$\begin{aligned} y_5 &= [[[x_s, x_t], x_t]x_t] = \\ &= -(1 - a_1b_0)b_0^2e_4 + (1 - a_1b_0)b_0b_4e_{n-1} + (b_2 - a_2b_0)b_0^2e_5 \\ &\quad - (b_2 - a_2b_0)b_0b_3e_{n-1} + (b_3 - a_3b_0)b_0^2e_4 - (b_3 - a_3b_0)b_0b_2e_{n-1} + \dots \\ &\quad + (b_{n-5} - a_{n-5}b_0)b_0^2e_{n-2} \\ &= (1 - a_1b_0)b_0^2e_4 + (b_2 - a_2b_0)b_0^2e_5 - (b_3 - a_3b_0)b_0^2e_5 \\ &\quad + (b_4 - a_4b_0)b_0^2e_7 - \dots - (b_{n-5} - a_{n-5}b_0)b_0^2e_{n-2} + \dots + ((1 - a_1b_0)b_0b \\ &\quad + (b_2 - a_2b_0)b_0b_3 - (b_3 - a_3b_0)b_2b_2 + \dots)b_0e_{n-1} \end{aligned}$$

$$\begin{aligned} y_6 &= [[[[x_s, x_t], x_t]x_t]x_t] = \\ &= -(1 - a_1b_0)b_0^3e_5 + (b_2 - a_2b_0)b_0^3e_4 - \dots - (b_{n-6} - a_{n-6}b_0)b_0^3e_{n-2} \\ &\quad + (b_2 - a_2b_0)b_0^2b_2e_{n-1} + (1 - a_1b_0)b_0^2b_2e_{n-1} \end{aligned}$$

$$\begin{aligned} y_7 &= [[[[[x_s, x_t], x_t]x_t]x_t]x_t] = \\ &= -(1 - a_1b_0)b_0^4e_6 + (b_2 - a_2b_0)b_0^4e_7 - \dots - (b_{n-7} - a_{n-7}b_0)b_0^4e_{n-2} \\ &\quad + (b_2 - a_2b_0)b_0^3b_2e_{n-1} + (1 - a_1b_0)b_0^2b_3e_{n-1} \end{aligned}$$

$$\begin{aligned} y_8 &= [[[[[[x_s, x_t], x_t]x_t]x_t]x_t]x_t] = \\ &= -(1 - a_1b_0)b_0^5e_7 + (b_2 - a_2b_0)b_0^5e_8 - \dots - (b_{n-8} - a_{n-8}b_0)b_0^5e_{n-2} \\ &\quad + (1 - a_1b_0)b_0^4b_3e_{n-1} \end{aligned}$$

Case 1. In this case, let  $b_0 \neq 0$ . We introduce the following definition

$$y_1 = x_s, \quad y_2 = x_t \quad y_3 = [x_s, x_t] \quad y_{n-1} = [[x_s, x_t] \dots x_t]$$

Let's assume the algebra has dimension  $n$ . In that case, the grading will look as follows:

$$y_1 \in V_1 \quad y_2 \in V_{k_t} \quad y_3 \in V_{k_s+k_t} \quad y_{n-1} \in V_{k_s+(n-3)k_t} \quad y_n \in V_n$$

Then, according to the property of the grading's compatibility,  $k_s = 1$

Let us assume:



$$k_s = 2, \alpha = 0, n$$

We can write our grading in the following form:

$$y_1 \in V_1 \quad y_2 \in V_2 \quad y_3 \in V_3 \quad y_{n-1} \in V_{n-1} \quad y_n \in V_n$$

$$V_1 + V_2 + V_3 + \dots + V_{n-1}$$

And check the conditions of the grading:

$$y_1 y_2 = V_1 V_2 \in V_3, \quad y_3 \in V_3$$

It satisfies the grading condition.

$$y_1 y_3 = [x_s [x_s x_t]]$$

$$= (1 - a_1 b_0) e_3 + (b_2 - a_2 b_0) e_4 - (b_3 - a_3 b_0) e_5 + \dots$$

$$+ (b_{n-4} - a_{n-4} b_0) e_{n-2} + (b_5 - a_5 b_0) a_1 e_{n-1} - (b_4 - a_4 b_0) a_2 e_{n-1}$$

$$+ (b_3 - a_3 b_0) a_3 e_{n-1} - (b_2 - a_2 b_0) a_4 e_{n-1} - (1 - a_1 b_0) a_5 e_{n-1}$$

This product does not satisfy the grading condition.

Case 2.

Let us assume the grading has dimension  $n-1$ .

2.1.  $k_s = 2, \alpha = 0, n-1$ . In this case, we obtain the following results:

$$y_1 \in V_1, \quad y_2 \in V_2, \quad y_3 \in V_3, \quad y_{n-1} \in V_{n-1}, \quad y_n \in V_n$$

$$y_1 y_3 = [x_s [x_s x_t]]$$

$$= (1 - a_1 b_0) e_3 + (b_2 - a_2 b_0) e_4 - (b_3 - a_3 b_0) e_5 + \dots$$

$$+ (b_{n-4} - a_{n-4} b_0) e_{n-2} + (b_5 - a_5 b_0) a_1 e_{n-1} - (b_4 - a_4 b_0) a_2 e_{n-1}$$

$$+ (b_3 - a_3 b_0) a_3 e_{n-1} - (b_2 - a_2 b_0) a_4 e_{n-1} - (1 - a_1 b_0) a_5 e_{n-1}$$

$$y_1 y_3 = V_2 V_3 \in V_5, \quad y_4 \in V_5$$

$$y_n \in V_5 \quad y_n = c_1 e_3 + c_2 e_{n-1}$$

$$y_2 y_n = (b_0 e_0 + b_1 e_1 + b_2 e_2 + \dots + b_{n-2} e_{n-2})(c_1 e_3 + c_2 e_{n-1}) = c_1 b_0 e_4 + b_4 c_1 e_{n-1}$$

Now, we will determine the unknown coefficient  $c_1$ .

$$c_1 = 1 - a_1 b_0$$

We encounter a contradiction:

$$y_2 y_n = V_1 V_5 \neq V_6$$

Therefore,  $k_s \neq 2$

2.2. Let be  $k_2 = 1$ . We will consider the following products:

$$y_1 y_3 = [x_s [x_s x_t]]$$

$$= (1 - a_1 b_0) e_3 + (b_2 - a_2 b_0) e_4 - (b_3 - a_3 b_0) e_5 + \dots$$

$$+ (b_{n-4} - a_{n-4} b_0) e_{n-2} + (b_5 - a_5 b_0) a_1 e_{n-1} - (b_4 - a_4 b_0) a_2 e_{n-1}$$

$$+ (b_3 - a_3 b_0) a_3 e_{n-1} - (b_2 - a_2 b_0) a_4 e_{n-1} - (1 - a_1 b_0) a_5 e_{n-1}$$

$$y_1 y_3 = V_2 V_3 \in V_3,$$

$$y_4 = -(1 - a_1 b_0) b_0 e_3 + (b_2 - a_2 b_0) b_0 e_4 - (b_3 - a_3 b_0) b_0 e_5 + \dots$$

$$+ (b_{n-4} - a_{n-4} b_0) b_0 e_{n-2} + (b_2 - a_2 b_0) a_3 e_{n-1} - (1 - a_1 b_0) a_4 e_{n-1}$$

$y_4 \in V_3$ , to satisfy the grading condition, we assume  $b_0 = 1$

$$y_1 y_4 = [x_s [x_s [x_s x_t]]]$$

$$= (1 - a_1 b_0) e_5 + (b_2 - a_2 b_0) e_4 - (b_3 - a_3 b_0) e_5 + \dots$$

$$+ (b_{n-5} - a_{n-5} b_0) e_{n-2} + (b_4 - a_4 b_0) a_1 e_{n-1} + \dots + (1 - a_1 b_0) a_4 e_{n-1}$$

$$y_1 y_4 = V_1 V_3 \in V_4, y_5 \in V_4$$

$k_s = 1$  satisfies the grading conditions.

$$\begin{aligned} y_1 &= e_0 - a_1 e_1 \\ y_3 &= (1 - a_1) e_2 \\ y_4 &= -(1 - a_1) e_3 \\ y_8 &= -(1 - a_1) e_7 + (1 - a_1) e_{n-1} \\ y &= (-1)^{-1} (1 - a_1) e_{n-1} \end{aligned}$$

Therefore, our grading has dimension  $n-1$  and takes the following form:

$$y_0, y_1 \in V_1, \quad y_3 \in V_2, \quad y_4 \in V_3, \quad y_n \in V_{n-1}$$

### CONCLUSION

In this paper, naturally graded  $n - 1$  length 2-filiform Lie algebras are determined and applied to determine the length of quasi-filiform Lie algebras. Using several references, the maximal-length 2-filiform Lie algebras were studied.

At the same time, in this master's thesis, several theorems concerning the  $n - 1$  length 2-filiform Lie algebras are presented. Additionally, the lengths of naturally graded 2-filiform Lie algebras were calculated, and among them, those with length equal to  $n - 1$  were identified. The final chapter of this dissertation is dedicated to identifying the even and odd-dimensional quasi-filiform Lie algebras with length  $n - 1$ .

First, the lengths of naturally graded odd-dimensional quasi-filiform Lie algebras were calculated, and it was determined that they have length  $n - 1$ . For this purpose, their decomposition into graded part spaces was used, and basis changes were applied to maximize the length of the grading, yielding the necessary results. In the second paragraph of the third chapter, the length of the naturally graded even-dimensional quasi-filiform Lie algebras was also calculated, and it was determined that the length of the given algebra is  $n - 1$ .

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